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LEIBNITZ-HAAR WAVELET COLLOCATION METHOD FOR THE NUMERICAL SOLUTION OF NONLINEAR FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT

In this work, we present a numerical solution of nonlinear fredholm integral equations using Leibnitz-Haar wavelet collocation method. Properties of haar wavelet and its operational matrix is utilized to convert into a system of algebraic equations, solving these equations using MATLAB to compute the required Haar coefficients. The numerical result of the proposed method is presented in comparison with the solutions given in the literature [3, 18 & 19] of the illustrative examples. Error analysis is worked out, which shows the efficiency of the method.

KEYWORDS: Operational Matrix, Leibnitz-Haar Wavelet Collocation Method (LHWCM), Nonlinear Fredholm Integral Equations.

INTRODUCTION

Integral equations find its applications in various fields of science and engineering. There are several numerical methods for approximating the solution of nonlinear Fredholm integral equations are known and many different basic functions have been used [1-7]. In numerical analysis solving integral equations are reducing it to a system of equations. There are various methods to solve integral equations such as Adomian decomposition method, successive substitutions, Laplace transformation method, Picard's method etc [22].

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [8-9]. Since from 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations, a detailed survey on these papers can be found in [10]. The solutions are often quite complicated and the advantages of the wavelet method get lost. Therefore any kind of simplification is welcome. One possibility for it is to make use of the Haar wavelets, which are mathematically the simplest wavelets. Haar wavelet methods are applied for different type of problems in [10-17]. Lepik et al. [18], Babolian et al. [19] and Aziz et al. [20] have applied the Haar wavelet method for solving nonlinear Fredholm integral equations. In the present work, a new approach for the numerical solution of nonlinear fredholm integral equations using Leibnitz-Haar wavelet collocation method is proposed.

The article is organized as follows: In Section 2, the properties of Haar wavelets and its operational matrix is given. Section 3 is devoted to the method of solution. In section 4, we report our numerical results and demonstrated the accuracy of the proposed scheme. Conclusion is discussed in section 5.

PROPERTIES OF HAAR WAVELETS

Haar wavelets

The scaling function $h_1(x)$ for the family of the Haar wavelet is defined as

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$$h_1(x)=\begin{cases}1 & for\ x\in[0,1)\\ 0 & otherwise\end{cases} \tag{1}$$

The Haar Wavelet family for $x\in[0,1)$ is defined as,

$$h_i(x)=\begin{cases}1 & for\ x\in[\alpha,\beta),\\ -1 & for\ x\in[\beta,\gamma),\\ 0 & elsewhere,\end{cases} \tag{2}$$

where $\alpha=\frac{k}{m},\quad \beta=\frac{k+0.5}{m},\quad \gamma=\frac{k+1}{m},$

where $m=2^l$, $l=0,1,...,J$, J is the level of resolution; and $k=0,1,...,m-1$ is the translation parameter. Maximum level of resolution is J. The index i in Eq. (2) is calculated using $i=m+k+1$. In case of minimal values $m=1,\ k=0$ then $i=2$. The maximal value of i is $N=2^{J+1}$.

Let us define the collocation points $x_j=\frac{j-0.5}{N}, j=1,2,...,N$, Haar coefficient matrix $H(i,j)=h_i(x_j)$ which has the dimension $N\times N$. For instance, $J=3\Rightarrow N=16$, then we have

$$\mathrm{H}(16,16)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1\\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1\\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

Operational Matrix of Haar Wavelet

The operational matrix *P* which is an N square matrix is defined by

$$P_{1,i}(x)=\int_0^x h_i(t)\,dt \tag{3}$$

often, we need the integrals

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$$P_{r,i}(x)=\underbrace{\int_A^x\int_A^x\ldots\int_A^x h_i(t)\,dt^r}_{r-times}=\frac{1}{(r-1)!}\int_A^x(x-t)^{r-1}h_i(t)\,dt \tag{4}$$

$r=1,2,...,n$ *and* $i=1,2,...,N.$

For $r=1,$ corresponds to the function $P_{1,i}(x)$, with the help of (2) these integrals can be calculated analytically, we get

$$P_{1,i}(x)=\begin{cases}x-\alpha & for\ \ x\in[\alpha,\beta)\\ \gamma-x & for\ \ x\in[\beta,\gamma)\\ 0 & Otherwise\end{cases} \tag{5}$$

$$P_{2,i}(x)=\begin{cases}\frac{1}{2}(x-\alpha)^2 & for\ \ x\in[\alpha,\beta)\\ \frac{1}{4m^2}-\frac{1}{2}(\gamma-x)^2 & for\ \ x\in[\beta,\gamma)\\ \frac{1}{4m^2} & for\ \ x\in[\gamma,1)\\ 0 & Otherwise\end{cases} \tag{6}$$

In general, the operational matrix of integration of r^{th} order is given as

$$P_{r,i}(x)=\begin{cases}\frac{1}{r!}(x-\alpha)^r & for\ \ x\in[\alpha,\beta)\\ \frac{1}{r!}\{(x-\alpha)^r-2(x-\beta)^r\} & for\ \ x\in[\beta,\gamma)\\ \frac{1}{r!}\{(x-\alpha)^r-2(x-\beta)^r+(x-\gamma)^r\} & for\ \ x\in[\gamma,1)\\ 0 & Otherwise\end{cases} \tag{7}$$

For instance, J=3 ⇒ N = 16, then we have

$$P_{1,i}(16,16)=\frac{1}{32}\begin{pmatrix}1&3&5&7&9&11&13&15&17&19&21&23&25&27&29&31\\1&3&5&7&9&11&13&15&15&13&11&9&7&5&3&1\\1&3&5&7&7&5&3&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&3&5&7&7&5&3&1\\1&3&3&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&3&3&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&3&3&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&1&3&3&1\\1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\end{pmatrix}$$

and

$$P_{2,i}(16,16)=\frac{1}{2048}\begin{pmatrix}
1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 289 & 361 & 441 & 529 & 625 & 729 & 841 & 961\\
1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 287 & 343 & 391 & 431 & 463 & 487 & 503 & 511\\
1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127\\
1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32\\
0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31\\
1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7
\end{pmatrix}$$

LEIBNITZ-HAAR WAVELET COLLOCATION METHOD (LHWCM)

In this section, we present a Leibnitz-Haar wavelet collocation method (LHWCM) for solving nonlinear Fredholm integral equation of the second kind,

$$u(x)=f(x)+\int_0^1 K(x,t,u(t))\,dt, \tag{8}$$

where *K(x, t, u(t))* is a nonlinear function defined on [0, 1]×[0, 1] are the known function *K(x, t, u(t))* is called the kernel of the integral equation while the unknown function *u(x)* represents the solution of the integral equation. The conversion of the integral equations into an equivalent differential equations. The conversion is achieved by using the well-known Leibnitz rule [22] for differentiation of integrals.

$$\text{Let,}\quad F(x)=\int_{g(x)}^{h(x)} K(x,t)u(t)\,dt \tag{9}$$

Then differentiation of the integral in (9) exists and is given by

$$F'(x)=\frac{dF}{dx}=K(x,h(x))(u(h(x)))\frac{dh(x)}{dx}-K(x,g(x))(u(g(x)))\frac{dg(x)}{dx}+\int_{g(x)}^{h(x)}\frac{\partial K(x,t)}{\partial x}u(t)\,dt \tag{10}$$

If $g(x)=0$ and $h(x)=1$, where 0 & 1 are fixed constants, then the Leibnitz rule (10) reduces to

$$F'(x)=\frac{dF}{dx}=\int_0^1\frac{\partial K(x,t)}{\partial x}u(t)\,dt \tag{11}$$

A numerical computation procedure is as follows:

Step 1: Differentiating (8) twice w.r.t *x*, using Leibnitz rule (10) we get,

$$u'(x)=f'(x)+F'(x) \tag{12}$$

$$u''(x)=f''(x)+F''(x) \tag{13}$$

Subject to initial conditions, $u(0)=\beta,\ u'(0)=\gamma$ (14)

Step 2: Applying Haar wavelet collocation method,

Let us assume that, $$u''(x) = \sum_{i=1}^{N} a_i h_i(x) \tag{15}$$

Step 3: By integrating (15) twice and using (14), we get (16) & (17),

$$u'(x) = \gamma + \sum_{i=1}^{N} a_i p_{1,i}(x) \tag{16}$$

$$u(x) = \beta + \gamma x + \sum_{i=1}^{N} a_i p_{2,i}(x) \tag{17}$$

Step 4: Substituting (15) – (17) in the differential equation (13), which reduces to the nonlinear system of *N* equations with *N* unknowns and then Newton's method can be used to find the Haar coefficients a_i, $i = 1, 2, ..., N.$ Substituting Haar coefficients in (17) to obtain the required approximate solutions of equation (8).

ILLUSTRATIVE EXAMPLES

In this section, we consider the some of the illustrative examples from the literature to demonstrate the capability of the method and error function is presented to verify the accuracy and efficiency of the numerical results:

$$Error\ function = E_{\max} = \left\| u_e(x_i) - u_a(x_i) \right\|_{\max} = \sqrt{\sum_{i=1}^{n} \left(u_e(x_i) - u_a(x_i) \right)^2}$$

where u_e and u_a are the exact and approximate solutions respectively.

Example 1. First**,** consider the Nonlinear Fredholm Integral equation [19],

$$u(x) + \int_0^1 e^{x-2t} \left[u(t)\right]^3 dt = e^{x+1}, \quad 0 \le x \le 1, \tag{18}$$

with initial conditions $u(0) = 1.$ Which has the exact solution $u(x) = e^x$.

Differentiating (18) w.r.t *x* and using Leibnitz rule (10), its equivalent differential equation (20),

$$u'(x) = e^{x+1} - \int_0^1 e^{x-2t} \left[u(t)\right]^3 dt \tag{19}$$

$$u'(x) - u(x) = 0 \tag{20}$$

Let us assume that, $$u'(x) = \sum_{i=1}^{N} a_i h_i(x) \tag{21}$$

integrating (21), we get

$$u(x) = 1 + \sum_{i=1}^{N} a_i p_{1,i}(x) \tag{22}$$

substituting (21)-(22) in the differential equation (20), we get the system of *N* equations with *N* unknowns

$$\sum_{i=1}^{N} a_i h_i(x) - \left(1 + \sum_{i=1}^{N} a_i p_{1,i}(x) \right) = 0 \tag{23}$$

solving (23) using Newton's Method to obtain Haar wavelet coefficients a_i's for *N* = 16 i.e., [1.7192 -0.4212 -0.1615 -0.2662 -0.0710 -0.0911 -0.1170 -0.1503 -0.0333 -0.0377 -0.0428 -0.0485 -0.0549 -0.0622 -0.0705 -0.0799]. Substituting a_i's, in (22) and obtained the required LHWCM solutions and is presented in table 1 & fig 1, in comparison with the exact and existing solutions. Error analysis is shown in table 2, which justifies the efficiency of the LHWCM.

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Table 1: Comparison of Exact and LHWCM of Example 1, for $N = 32$.

x	Exact	LHWCM	Method [19]	Error (LHWCM)	Error (Method [19])
0.1	1.105170918	1.105314848	1.107217811	1.4e-04	2.0e-03
0.2	1.221402758	1.221571768	1.218102916	1.6e-04	3.3e-03
0.3	1.349858808	1.350056580	1.341165462	1.9e-04	8.7-03
0.4	1.491824698	1.492055414	1.474918603	2.3e-04	1.6e-02
0.5	1.648721271	1.648989674	1.667402633	2.6e-04	1.8e-02
0.6	1.822118800	1.822430264	1.833861053	3.1e-04	1.1e-02
0.7	2.013752707	2.014113322	2.016679830	3.6e-04	2.9e-03
0.8	2.225540928	2.225957586	2.217456630	4.1e-04	8.1e-03
0.9	2.459603111	2.460083612	2.437978177	4.8e-04	2.1e-02

Table 2: Maximum error analysis of Example 1.

N	$E_{\max}(LHWCM)$
4	3.0e-02
8	8.1e-03
16	2.1e-03
32	5.4e-04
64	1.3e-04
128	3.4e-05

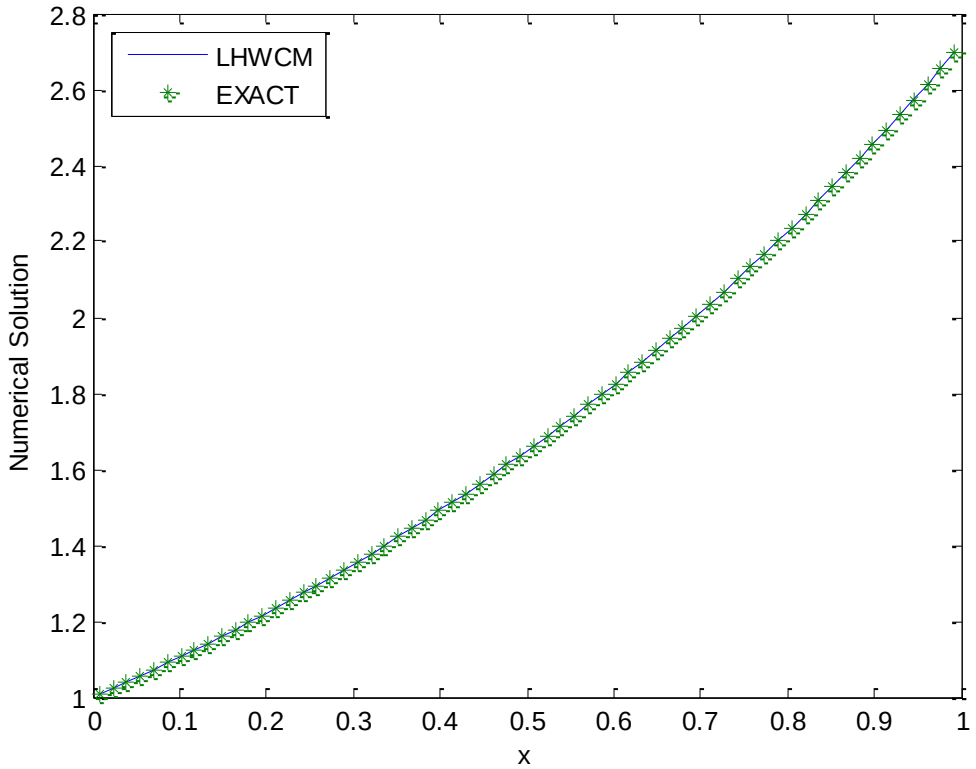

Fig 1: Comparison of LHWCM with exact solution for N=64 of Example 1.

Example 2. Next, consider the Nonlinear Fredholm Integral equation [19],

$$u(x)-\int_0^1 xt[u(t)]^3\,dt = e^x - \frac{(1+2e^3)x}{9}, \qquad 0\le x\le 1 \tag{24}$$

with initial conditions $u(0)=1, u'(0)=1$. Which has the exact solution $u(x)=e^x$.
Differentiating (24) w.r.t x, using Leibnitz rule (10) which reduces to the differential equation (26),

$$u'(x) = e^x - \frac{(1+2e^3)}{9} + \int_0^1 t[u(t)]^3\,dt, \tag{25}$$

$$u''(x) - e^x = 0 \tag{26}$$

Let us assume that, $$u''(x) = \sum_{i=1}^{N} a_i h_i(x) \tag{27}$$

integrating (27) twice,

$$u'(x) = \sum_{i=1}^{N} a_i p_{1,i}(x) + 1 \tag{28}$$

$$u(x) = \sum_{i=1}^{N} a_i p_{2,i}(x) + x + 1 \tag{29}$$

substituting (27)-(29) in (26), we get the system of N equations with N unknowns.

$$\sum_{i=1}^{N} a_i h_i(x) - e^x = 0 \tag{30}$$

solving (30) using Newton's Method to obtain Haar wavelet coefficients a_i's for N = 16 i.e., [1.7180 -0.4208 -0.1613 -0.2660 -0.0709 -0.0910 -0.1169 -0.1501 -0.0333 -0.0377 -0.0427 -0.0484 -0.0549 -0.0622 -0.0704 -0.0798]. Substituting a_i's, in (29) and obtained the required LHWCM solutions and is presented in table 3 & fig 2 in comparison with the exact and existing solutions. Error analysis is shown in table 4, which justifies the efficiency of the LHWCM.

Table 3: Comparison of Exact and LHWCM of Example 2, for N = 32.

x	Exact	LHWCM	Method [19]	Error (LHWCM)	Error (Method [19])
0.1	1.105170918	1.105179274	1.114627560	8.3e-06	9.5e-03
0.2	1.221402758	1.221419913	1.226758840	1.7e-05	5.4e-03
0.3	1.349858808	1.349885259	1.345458179	2.6e-05	4.4e-03
0.4	1.491824698	1.491860995	1.480202470	3.6e-05	1.1e-02
0.5	1.648721271	1.648768022	1.671769819	4.6e-05	2.3e-02
0.6	1.822118800	1.822176677	1.838854903	5.7e-05	1.6e-02
0.7	2.013752707	2.013822451	2.022118086	6.9e-05	8.4e-03
0.8	2.225540928	2.225623359	2.223139077	8.2e-05	2.4e-03
0.9	2.459603111	2.459699136	2.443684898	9.6e-05	1.5e-02

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Table 4: Maximum error analysis of Example 2.

N	$E_{\max}(LHWCM)$
4	5.9e-03
8	1.6e-03
16	4.2e-04
32	1.0e-04
64	2.7e-05
128	6.8e-06

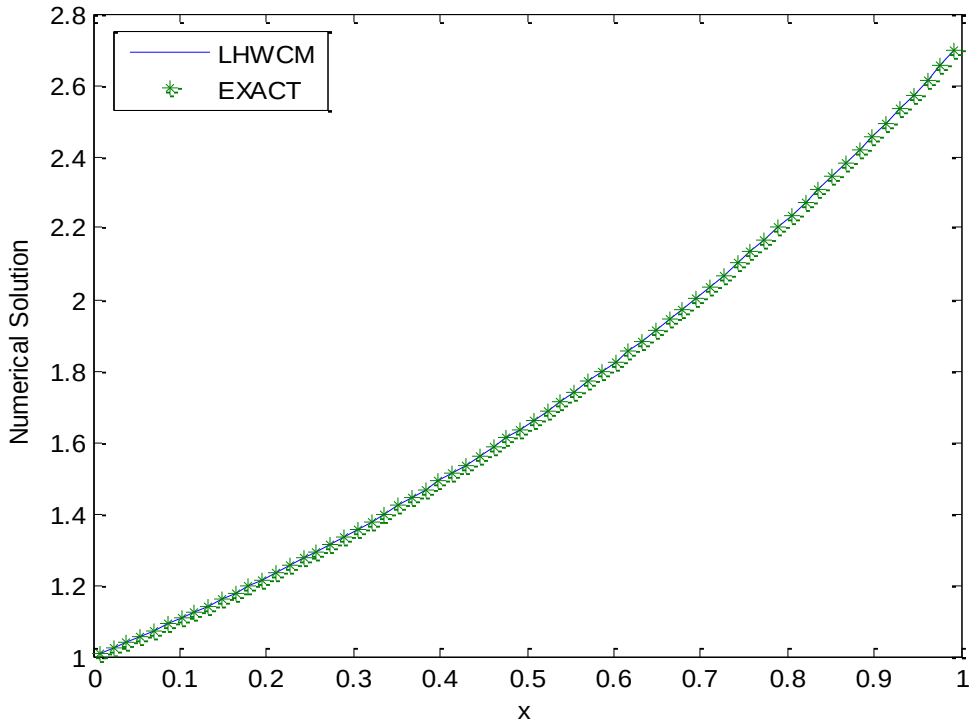

Fig 2: Comparison of LHWCM with exact solution for N=64 of Example 2.

Example 3. Now, consider the Nonlinear Integral equation [18],

$$u(x) = -x^2 - \frac{x}{3}(2\sqrt{2}-1) + 2 + \int_0^1 xt\sqrt{u(t)}\ dt, \qquad 0 \le x \le 1, \tag{31}$$

with initial conditions $u(0) = 2,\ u'(0) = 0.$ Which has the exact solution $u(x) = 2 - x^2$.

Differentiating (31) w.r.t x, and using Leibnitz rule (10) which reduces to the differential equation (33),

$$u'(x) = -2x - \frac{1}{3}(2\sqrt{2}-1) + \int_0^1 t\sqrt{u(t)}\ dt, \tag{32}$$

$$u''(x) + 2 = 0 \tag{33}$$

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Let us assume that, $$u''(x) = \sum_{i=1}^{N} a_i h_i(x) \quad (34)$$

integrating (34) twice,

$$u'(x) = \sum_{i=1}^{N} a_i p_{1,i}(x) \quad (35)$$

$$u(x) = \sum_{i=1}^{N} a_i p_{2,i}(x) + 2 \quad (36)$$

substituting (34)-(36) in (33), we get the system of N equations with N unknowns.

$$\sum_{i=1}^{N} a_i h_i(x) + 2 = 0 \quad (37)$$

solving (37) using Newton's Method to obtain Haar wavelet coefficients a_i's for N = 16 i.e., [-2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]. Substituting a_i's, in (36) and obtained the required LHWCM solutions, which gives the exact solutions. Fig 3 shows the comparison of approximate solutions with the exact solutions, which justifies the efficiency of the LHWCM.

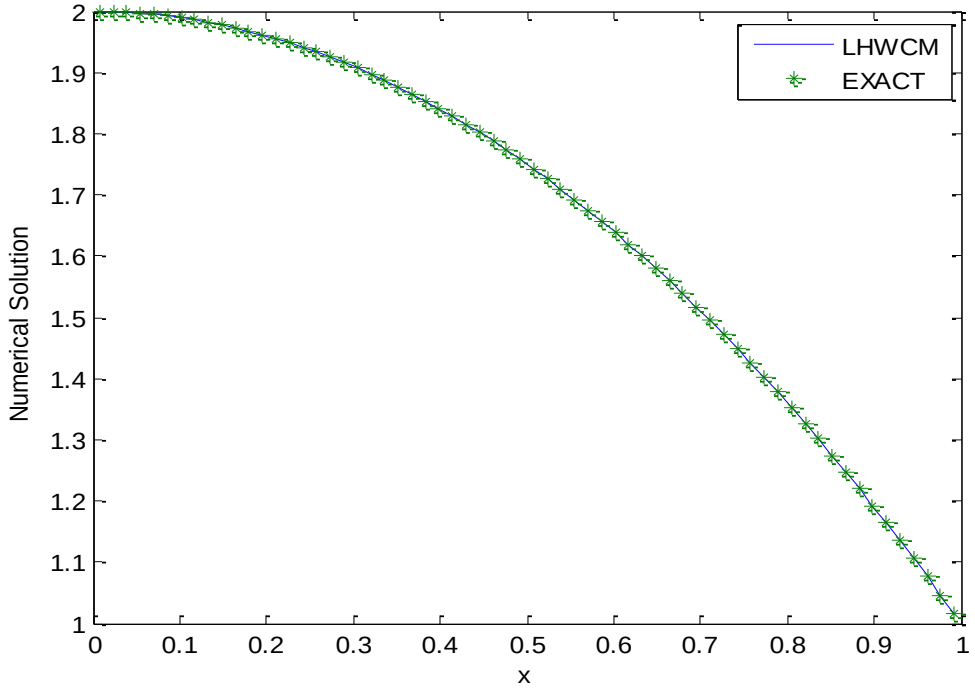

Fig 3: Comparison of LHWCM with exact solution for N=64 of Example 3.

CONCLUSION

The aim of this paper, numerical solution of nonlinear fredholm integral equations using Leibnitz-Haar wavelet collocation method. Using lebnitz rule, converts integral equations into differential equations with initial conditions. The Haar wavelet function and its operational matrix were employed to solve the resultant differential equations. Our present method avoids the tedious work, it minimizes the computational calculus and supplies quantitatively reliable results. The results obtained by the proposed method have been compared with the existing methods and the exact solutions. The illustrative examples have been included to justify the efficiency and which confirms plausibility of new technique.

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